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LETTER TO THE EDITOR

Two simple proofs of the Kochen–Specker theorem

Asher Peres†

Department of Physics, Technion-Israel Institute of Technology, 32000 Haifa, Israel

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Abstract. A new proof of the Kochen–Specker theorem uses 33 rays, instead of 117 in the original proof. If the number of dimensions is increased from 3 to 4, only 24 rays are needed.

The Kochen–Specker (1967) theorem is of fundamental importance for quantum theory. It asserts that, in a Hilbert space of dimension $\geqslant 3$, it is impossible to associate definite numerical values, 1 or 0, with every projection operator P_m, in such a way that, if a set of *commuting* P_m satisfies $\Sigma\, P_m = 1$, the corresponding values $v(P_m)$ will also satisfy $\Sigma\, v(P_m) = 1$. The thrust of this theorem is that any purported cryptodeterministic theory which would attribute a definite result to each quantum measurement, and still reproduce the statistical properties of quantum theory, must necessarily be *contextual.* Namely, if three operators A, B and C satisfy $[A, B] = [A, C] = 0$ and $[B, C] \neq 0$, the result of a measurement of A cannot be independent of whether A is measured alone, or together with B, or together with C (Bell 1966, Redhead 1987).

The proof of the theorem runs as follows. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ be a complete set of orthonormal vectors. The N matrices $P_m = \boldsymbol{u}_m \boldsymbol{u}_m^\dagger$ are projection operators on the vectors $\boldsymbol{u}_m$. These matrices commute and satisfy $\Sigma\, P_m = 1$. There are N different ways of associating the value 1 with one of these matrices (that is, with one of the vectors $\boldsymbol{u}_m$), and the value 0 with the $N-1$ others. Consider now several distinct orthogonal bases, which may share some of their unit vectors. Assume that if a vector is a member of more than one basis, the value (1 or 0) associated with that vector is the same, irrespective of the choice of the other basis vectors. This assumption leads to a contradiction, as shown by Kochen and Specker (1967) for a particular set of 117 vectors in $\boldsymbol{R}^3$. An earlier proof by Bell (1966) involved a *continuum* of vector directions. Both proofs were motivated by the Gleason (1957) theorem.

Although this result has a fundamental importance, its lengthy proof is seldom given in textbooks—a notable exception being Redhead (1987). Over the years, there were many attempts to reduce the number 117, with meagre results. However, very recently, Conway and Kochen (private communication) found a set of 31 vectors having the same property. The directions of these vectors are obtained by connecting the origin with suitably selected points whose coordinates are small integers. Later in this letter, I shall introduce a different set in $\boldsymbol{R}^3$, with 33 vectors belonging to 16 distinct bases. That set enjoys many symmetries making the proof of the theorem remarkably brief. I will then give a similar proof in $\boldsymbol{R}^4$, using only 24 vectors.

† Bitnet: phr30pe@technion.

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In these proofs, I shall use the word *ray*, rather than *vector*, because only directions are relevant. The length of the vectors never plays any role, and it is in fact convenient to let that length exceed 1. This does not affect orthogonality, and the algebra becomes easier. To further simplify the discourse, rays associated with the values 1 and 0 will be called green and red, respectively (as in traffic lights, green = yes, red = no).

The 33 rays are those for which the squares of the direction cosines are one of the combinations

$$0+0+1=0+\tfrac{1}{2}+\tfrac{1}{2}=0+\tfrac{1}{3}+\tfrac{2}{3}=\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{2} \tag{1}$$

and all permutations thereof. These rays can be obtained by connecting the origin to various points, labelled xyz, where x, y and z can be: 0, 1, $\bar{1}$ (this symbol stands for -1), 2 (means $\sqrt{2}$), and $\bar{2}$ (means $-\sqrt{2}$). For example the ray $\bar{1}02$ connects the origin to the point $(-1, 0, \sqrt{2})$. The squares of the direction cosines of that ray are $\frac{1}{3}$, 0 and $\frac{2}{3}$. Opposite rays, such as $\bar{1}02$ and $10\bar{2}$, are counted only once.

An important property of this set of rays is its invariance under interchanges of the x, y and z axes, and under a reversal of the direction of each axis. This allows to assign arbitrarily—without loss of generality—values 1 and 0 to some of the rays, because any other assignment would be equivalent to renaming the axes, or reversing one of them. For example, one can impose that ray 001 is green, while 100 and 010 are red.

The proof of the KS theorem entirely holds in table 1. In each line, the first ray, printed in boldface characters, is green. The second and third rays form, together with the first one, an orthogonal triad. Therefore they are red. Additional rays listed in the same line are also orthogonal to its first ray, therefore they too are red (only the rays that will be needed for further work are listed). When a red ray is printed in italic characters, this means that it is an 'old' ray, that was already found red in a preceding line. The choice of colours for the new rays appearing in each line is explained in the table itself.

Table 1. Proof of KS theorem in three dimensions.

Orthogonal triad			Other rays		The first ray is green because of
001	100	010	110	$1\bar{1}0$	choice of z axis
101	$\bar{1}01$	*010*			choice of x vs $-x$
011	$0\bar{1}1$	*100*			choice of y vs $-y$
$\mathbf{1\bar{1}2}$	$\bar{1}12$	*110*	$\bar{2}01$	021	choice of x vs y
102	$\mathit{\bar{2}01}$	*010*	$\bar{2}11$		orthogonality to second and third rays
211	$\mathit{0\bar{1}1}$	$\mathit{\bar{2}11}$	$\bar{1}02$		orthogonality to second and third rays
201	*010*	$\mathit{\bar{1}02}$	$\bar{1}\bar{1}2$		orthogonality to second and third rays
112	$\mathit{1\bar{1}0}$	$\mathit{\bar{1}\bar{1}2}$	$0\bar{2}1$		orthogonality to second and third rays
012	*100*	$\mathit{0\bar{2}1}$	$1\bar{2}1$		orthogonality to second and third rays
121	$\mathit{\bar{1}01}$	$\mathit{1\bar{2}1}$	$0\bar{1}2$		orthogonality to second and third rays

The first, fourth and last lines contain rays 100, 021 and $0\bar{1}2$, respectively. These three rays are red and mutually orthogonal: this is the KS contradiction. It can be shown that if a single ray is deleted from the set of 33, the contradiction disappears. It is so even if the deleted ray is not explicitly listed in table 1. This is because the removal of one ray breaks the symmetry of the set and therefore necessitates the examination of alternative choices. The proof that a contradiction can then be avoided is not quite as simple as table 1.

The physical interpretation of the KS theorem in $\boldsymbol{R}^3$ is well known. Each P_m projection operator can be written as $1-(\boldsymbol{m}\cdot\boldsymbol{J})^2$, where $\boldsymbol{m}$ is a unit vector and $\boldsymbol{J}$ is the angular momentum operator for a spin 1 particle. If $\boldsymbol{m}$ and $\boldsymbol{n}$ are orthogonal vectors, the operator

$$K=(\boldsymbol{m}\cdot\boldsymbol{J})^2-(\boldsymbol{n}\cdot\boldsymbol{J})^2 \tag{2}$$

has eigenvalues -1, 0 and 1. A direct measurement of that operator is technically possible (Swift and Wright 1980) and determines the 'colours' of the triad $\boldsymbol{m}$, $\boldsymbol{n}$ and $\boldsymbol{m}\times\boldsymbol{n}$. The 16 different triads correspond to 16 different operators like K, of which anyone (but only one) can be actually measured. The results of the other measurements are counterfactual—and contradictory.

With the same notations as above, the 24 rays, labelled *wxyz*, are 1000, 1100, $1\bar{1}00$, 1111, $111\bar{1}$, $11\bar{1}\bar{1}$, and all permutations thereof (opposite rays are counted only once). This set is invariant under interchanges of the *w*, *x*, *y* and *z* axes, and under a reversal of the direction of each axis.

Table 2 proves the KS theorem for this case. The conventions are the same as for table 1. The first line is obtained by labelling the green axis *w*; the second and third lines, by appropriate choices of the directions of the other axes. The green rays in the fourth and fifth lines are determined by orthogonality to three red rays. It is then found that the first, third and fifth lines contain rays 0110, $01\bar{1}0$, $100\bar{1}$ and 1001. These four rays are red and mutually orthogonal: again a contradiction. Note that all 24 rays appear explicitly in table 2. If a single one is deleted, the contradiction can be avoided.

Table 2. Proof of KS theorem in four dimensions.

Orthogonal tetrad				Other rays orthogonal to first one					
1000	0100	0010	0001	0011	$001\bar{1}$	0101	$010\bar{1}$	0110	$01\bar{1}0$
1100	$1\bar{1}00$	0011	$001\bar{1}$	$1\bar{1}1\bar{1}$	$1\bar{1}\bar{1}1$	$1\bar{1}11$	$\bar{1}111$		
1111	$11\bar{1}\bar{1}$	$1\bar{1}1\bar{1}$	$1\bar{1}\bar{1}1$	$10\bar{1}0$	$100\bar{1}$				
1010	$10\bar{1}0$	0101	$010\bar{1}$	$11\bar{1}1$					
$111\bar{1}$	$11\bar{1}1$	$1\bar{1}11$	$\bar{1}111$	1001					

There is an important difference between the two set of rays: the 24 rays in $\boldsymbol{R}^4$ form 6 *disjoint* orthogonal tetrads. Each ray belongs to a *single* tetrad. From the four projection operators corresponding to each tetrad, one obtains:

$$(P_1+P_2-P_3-P_4)(P_1-P_2+P_3-P_4)(P_1-P_2-P_3+P_4)=1. \tag{3}$$

Each parenthesis in the left hand side is an operator with eigenvalues 1, 1, -1 and -1. These three operators are a *complete* set of commuting operators. The choice of the 'green' P_m determines the values (1 or -1) associated with each one of the three operators, and the product of these values is 1.

The condition for existence of a KS contradiction can now be rephrased in terms of these triads of commuting operators, whose product is 1 (or -1). Let a number, 1 or -1, be associated with each operator, in such a way that the product of these three numbers is equal to the product of the operators in each triad. Moreover, let each operator belong to several triads, and assume that it has the same value in all the triads in which it appears. A contradiction may arise.

The simplest way of seeing this is by means of a concrete example. Consider a pair of spin-$\frac{1}{2}$ particles. In the square array

$$\begin{array}{ccc} 1\otimes\sigma_z & \sigma_z\otimes 1 & \sigma_z\otimes\sigma_z \\ \sigma_x\otimes 1 & 1\otimes\sigma_x & \sigma_x\otimes\sigma_x \\ \sigma_x\otimes\sigma_z & \sigma_z\otimes\sigma_x & \sigma_y\otimes\sigma_y \end{array} \qquad (4)$$

each row and each column is a triad of commuting operators, as described above. The KS contradiction is due to the fact that the product of the three operators in each row or column is $1\otimes 1$, except those of the third column, whose product is $-1\otimes 1$ (Mermin 1990). It is obviously impossible to associate numerical values, 1 or ± 1, to each one of these nine operators, obeying the same multiplication rule.

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